UNIQUENESS OF $\mathbb{CP}^n$

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Abstract. We give an exposition of a theorem of Hirzebruch, Kodaira and Yau which proves the uniqueness of the Kähler structure of complex projective space, and of Yau's resolution of the Severi Conjecture.

1. Introduction

It is a classical result in complex analysis that every simply connected closed Riemann surface is biholomorphic to the projective line $\mathbb{CP}^1$. The purpose of this note is to explain in detail two higher-dimensional generalizations of this fact.

Theorem 1.1 (Hirzebruch, Kodaira [4], Yau [14]). *If a Kähler manifold M is homeomorphic to $\mathbb{CP}^n$ then M is biholomorphic to it.*

More precisely, Hirzebruch and Kodaira proved this for all n odd, leaving open the case of n even which was finally solved by Yau. Also, Hirzebruch and Kodaira assumed that M is diffeomorphic to $\mathbb{CP}^n$, and this was relaxed to homeomorphic after work of Novikov. When $n = 2$, a stronger result holds, which was known as the Severi Conjecture [13], and was solved by Yau:

Theorem 1.2 (Yau [14]). *If a compact complex surface M is homotopy equivalent to $\mathbb{CP}^2$ then it is biholomorphic to it.*

A brief outline of the proofs of these theorems is the following. From the assumptions, using the Hirzebruch-Riemann-Roch theorem, one deduces that either M is Fano (i.e. $c_1(M)$ can be represented by a Kähler metric) or else the canonical bundle K_M is positive (i.e. $-c_1(M)$ can be represented by a Kähler metric). The second case can only arise when n is even. When M is Fano a geometric argument shows that M is biholomorphic to $\mathbb{CP}^n$, which settles the case when n is odd. On the other hand, when K_M is positive then a key inequality between Chern numbers holds, as shown by Yau. Furthermore, in our case we have that equality holds, and this implies that M is biholomorphic to the unit ball in $\mathbb{C}^n$, which is absurd because M is compact.

The details are presented in Section 2, mostly following the original sources (together with a small simplification of part of the argument from

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[7]), and in Section 3 we discuss a natural conjectural extension of these theorems, and how it is related to another well-known open problem.

2. Proofs of the main results

Proof of Theorem 1.1. The fact that M is Kähler gives us the Hodge decomposition on cohomology, which we will use repeatedly. From the hypothesis we see that

$$H^2(M,\mathbb{Z}) \cong \mathbb{Z}, \quad H^1(M,\mathbb{C}) \cong 0 \cong H^{0,1}(M),$$

$$H^2(M,\mathbb{C}) \cong \mathbb{C} \cong H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M),$$

and since $H^{2,0}(M) \cong H^{0,2}(M)$, we see that they are both zero, while $H^{1,1}(M) \cong \mathbb{C}$. Thanks to the vanishing of $H^{0,1}(M)$ and $H^{0,2}(M)$, the exponential exact sequence gives that the first Chern class map

$$c_1 : \mathrm{Pic}(M) \to H^2(M,\mathbb{Z}) \cong \mathbb{Z},$$

is an isomorphism, where as usual the Picard group $\mathrm{Pic}(M)$ is the group of isomorphism classes of holomorphic line bundles on M.

Lemma 2.1. *M is projective and its holomorphic Euler characteristic satisfies*

$$\chi(M,\mathcal{O}) := \sum_{p=0}^{n} (-1)^p \dim H^{0,p}(M) = 1.$$

Proof. Choose a Kähler form $\tilde{\omega}$ on M. Its cohomology class $[\tilde{\omega}]$ lies in $H^2(M,\mathbb{R}) \cong \mathbb{R}$ so we can rescale $\tilde{\omega}$ to get another Kähler form ω whose cohomology class generates $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$. We have that $\int_M \omega^n > 0$ because this equals $n!$ times the total volume of M measured using the Kähler metric ω. On $\mathbb{CP}^n$ a generator α of $H^2(M,\mathbb{Z})$ satisfies $\langle \alpha^{\smile n}, [\mathbb{CP}^n] \rangle = \pm 1$, and since ω is Kähler we have that $\int_M \omega^n = 1$. Since c_1 is an isomorphism, there exists $L \to M$ a holomorphic line bundle whose first Chern class is $[\omega]$. If h is a smooth Hermitian metric on the fibers of L then its curvature form γ is a closed real $(1,1)$ form cohomologous to $c_1(L) = [\omega]$. By the $\partial\overline{\partial}$-Lemma, which holds because M is Kähler, there is a smooth real-valued function ψ on M such that $\omega = \gamma + \sqrt{-1}\partial\overline{\partial}\psi$. The Hermitian metric $\tilde{h} = e^{-\psi}h$ on L then has curvature form equal to ω, and so L is a positive line bundle. Thanks to the Kodaira Embedding Theorem [5, Proposition 5.3.1], L is ample and the manifold M is projective.

Since $\int_M \omega^n \neq 0$, it follows that the classes $[\omega^k] \in H^{k,k}(M)$ are nonzero for $1 \leq k \leq n$, and as above the Hodge decomposition implies that $H^{p,q}(M) = 0$ if $p \neq q$. This gives that the holomorphic Euler characteristic of M satisfies $\chi(M,\mathcal{O}) = 1$. □

Recall the following definition: if $F \to M$ is a real vector bundle, then its Pontrjagin classes are defined to be $p_i(F) = (-1)^i c_{2i}(F \otimes \mathbb{C}) \in H^{4i}(M,\mathbb{Z})$, where c_{2i} denotes the $(2i)^{th}$ Chern class of the complex vector bundle $F \otimes \mathbb{C}$.

If $F = TM$ we just write $p_i(M)$. Now we need the following theorem, which we will quote without proof.

Theorem 2.2 (Novikov [12]). *The rational Pontrjagin classes of a closed smooth manifold are invariant under homeomorphism.*

Here the rational Pontrjagin classes are just the images of $p_i(M)$ under the natural map $H^{4i}(M, \mathbb{Z}) \to H^{4i}(M, \mathbb{Q})$. Since our manifold M has torsion-free integral cohomology, we obtain in our case the invariance of the integral Pontrjagin classes. In particular if $f : M \to \mathbb{CP}^n$ is the given homeomorphism, then $f^* p_i(\mathbb{CP}^n) = p_i(M)$ for all i. Notice that if f is assumed to be a diffeomorphism then this is obvious since $f^*(T\mathbb{CP}^n) \cong TM$ is an isomorphism of real vector bundles, which induces an isomorphism of complex vector bundles $f^*(T\mathbb{CP}^n \otimes \mathbb{C}) \cong TM \otimes \mathbb{C}$ which therefore preserves the Chern classes, so we do not need Novikov's theorem in that case. On the other hand, it is in general false that $f^* c_i(\mathbb{CP}^n) \cong c_i(M)$ when f is a diffeomorphism, which is why we are forced to work with Pontrjagin classes instead of Chern classes.

Lemma 2.3. *The holomorphic Euler characteristic of M satisfies*

$$\chi(M, \mathcal{O}) = \int_M e^{\frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)} \right)^{n+1}. \tag{2.1}$$

Proof. If H denotes the hyperplane class on $\mathbb{CP}^n$ then it is well-known (see e.g. [11, Example 15.6]) that

$$p_i(\mathbb{CP}^n) = \binom{n+1}{i} H^{2i},$$

for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. Moreover the fact that f is a homeomorphism implies that $f^* H$ is a generator of $H^2(M, \mathbb{Z})$ and so $f^* H = \pm[\omega]$. Putting these together we get

$$p_i(M) = \binom{n+1}{i} [\omega^{2i}], \tag{2.2}$$

for $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. The Hirzebruch-Riemann-Roch Theorem [3, Theorem 20.3.2] says that for any holomorphic line bundle F on M we have

$$\chi(M, F) := \sum_{p \geq 0} (-1)^p \dim H^p(M, F) = \int_M e^{c_1(F)} \mathrm{Td}(M),$$

where $\mathrm{Td}(M)$ is the Todd genus of M. This is defined in terms of the Chern classes of M, but since in our case we only know the Pontrjagin classes of M, we need to express $\mathrm{Td}(M)$ as much as possible in terms of these. To do this, we use the identity [3, p.150, (6*)]

$$\mathrm{Td}(M) = e^{\frac{c_1(M)}{2}} \hat{A}(M),$$

where the $\hat{A}$ genus of M is defined as follows (see [3] for details). We formally write

$$\sum_{j\geq 0} p_j(M)x^j = \prod_{j\geq 1}(1+\gamma_j x),$$

for some symbols γ_j, and let

$$\hat{A}(M) = \prod_{j\geq 0} \frac{\sqrt{\gamma_j}/2}{\sinh(\sqrt{\gamma_j}/2)},$$

which is therefore a polynomial in the Pontrjagin classes $p_j(M)$. Taking $F = \mathcal{O}$ in the Hirzebruch-Riemann-Roch formula (where $\mathcal{O}$ is the trivial line bundle) gives

$$\chi(M,\mathcal{O}) = \int_M e^{\frac{c_1(M)}{2}} \hat{A}(M).$$

Now thanks to (2.2) we have

$$\sum_{j\geq 0} p_j(M)x^j = (1+[\omega^2]x)^{n+1},$$

which gives $\gamma_1 = \cdots = \gamma_{n+1} = [\omega^2]$ and $\gamma_j = 0$ for $j > n+1$. Thus, we obtain the key identity (2.1). $\square$

In order to proceed with the proof, we need to determine $c_1(M)$.

Lemma 2.4. *We have that $c_1(M)$ equals either $(n+1)[\omega]$ or $-(n+1)[\omega]$, with the latter only possibly occurring when n is even.*

Proof. The reduction mod 2 of $c_1(M)$ is the second Stiefel-Whitney class $w_2(M) \in H^2(M,\mathbb{Z}_2)$, which is a topological invariant. Hence it is equal to $w_2(\mathbb{CP}^n)$ which is $c_1(\mathbb{CP}^n)$ mod 2, that is $n+1$ mod 2. On the other hand since $c_1(M)$ and $[\omega]$ both belong to $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$, we have $c_1(M) = \lambda[\omega]$ for some $\lambda \in \mathbb{Z}$, and so $\lambda = n+1+2s$ for some $s \in \mathbb{Z}$. From Lemma 2.3 we get

$$\chi(M,\mathcal{O}) = \int_M e^{\frac{n+1+2s}{2}\omega} \left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1} = \int_M e^{s\omega} \left(\frac{\omega}{1-e^{-\omega}}\right)^{n+1},$$

using the identity

$$\frac{x}{1-e^{-x}} = e^{\frac{x}{2}} \frac{x/2}{\sinh(x/2)}.$$

Since $\int_M \omega^n = 1$, and the integrals over M of all other powers of ω are zero by definition, this means that $\chi(M,\mathcal{O})$ equals the coefficient of x^n in the power series expansion of

$$e^{sx}\left(\frac{x}{1-e^{-x}}\right)^{n+1}.$$

Following [4] we give two different ways of calculating this coefficient. The first method uses residues, and more precisely the fact that if we define a holomorphic function F by

$$F(z) = e^{sz}\left(\frac{z}{1-e^{-z}}\right)^{n+1},$$

then Cauchy's integral formula shows that the coefficient that we are interested in equals the contour integral

$$\frac{1}{2\pi\sqrt{-1}}\oint \frac{F(z)}{z^{n+1}}dz = \frac{1}{2\pi\sqrt{-1}}\oint \frac{e^{sz}}{(1-e^{-z})^{n+1}}dz,$$

where the countour is a small circle around the origin, with counterclockwise orientation. Since the power series expansion of $1-e^{-z}$ at $z=0$ starts with z, this function is a local biholomorphism near the origin, so we can change variable $y = 1-e^{-z}$ near 0 and rewrite our contour integral as

$$\frac{1}{2\pi\sqrt{-1}}\oint \frac{1}{(1-y)^{(s+1)}y^{n+1}}dz,$$

where the contour is again a small circle around the origin. By the Residue theorem this integral equals the residue of the function $\frac{1}{(1-y)^{(s+1)}y^{n+1}}$ at 0, which is the coefficient of y^n in the Taylor expansion of $(1-y)^{-s-1}$ at 0. Expanding this function, we finally obtain that our desired coefficient equals

$$\binom{n+s}{n} = \frac{(n+s)(n+s-1)\cdots(s+1)}{n!},$$

where we allows $s<0$.

The second way to calculate this coefficient is as follows: by Hirzebruch-Riemann-Roch again, this coefficient equals

$$\begin{aligned}\int_{\mathbb{CP}^n} e^{sH}\left(\frac{H}{1-e^{-H}}\right)^{n+1} &= \int_{\mathbb{CP}^n} e^{sH}e^{\frac{n+1}{2}H}\left(\frac{H/2}{\sinh(H/2)}\right)^{n+1}\\ &= \int_{\mathbb{CP}^n} e^{c_1(\mathcal{O}(s))}e^{\frac{c_1(\mathbb{CP}^n)}{2}}\hat{A}(\mathbb{CP}^n)\\ &= \chi(\mathbb{CP}^n, \mathcal{O}(s)),\end{aligned}$$

and it is well-known (see e.g. [5, Example 5.2.5]) that $\chi(\mathbb{CP}^n, \mathcal{O}(s))$ equals $\binom{n+s}{n}$. So, using either of the two methods, we conclude that

$$\chi(M, \mathcal{O}) = \binom{n+s}{n}.$$

Since $\chi(M,\mathcal{O}) = 1$ by Lemma 2.1, we get that $\binom{n+s}{n} = 1$, which can be rewritten as

$$n! = (s+n)\cdots(s+1).$$

If n is odd this implies that $s=0$, while if n is even, s is either 0 or $-n-1$. But we saw that $c_1(M) = (n+1+2s)[\omega]$ and so if n is odd we

get $c_1(M) = (n+1)[\omega]$, while if n is even, $c_1(M)$ is either $(n+1)[\omega]$ or $-(n+1)[\omega]$. $\square$

Assume first that $c_1(M) = (n+1)[\omega]$, which implies that M is a Fano manifold (i.e. there is a Kähler metric in $c_1(M)$). Then $c_1(K_M) = -c_1(M) = -(n+1)c_1(L)$ and so $K_M = -(n+1)L$, since the map c_1 is an isomorphism. Then Serre duality gives $H^k(M,L) \cong H^{n-k}(M, K_M - L)$ and $K_M - L = -(n+2)L$ is negative, so $H^k(M,L) = 0$ if $k > 0$ by Kodaira vanishing. Hence using Hirzebruch-Riemann-Roch again we get

$$\begin{aligned}\dim H^0(M,L) = \chi(M,L) &= \int_M e^{c_1(L)+\frac{c_1(M)}{2}} \left(\frac{\omega/2}{\sinh(\omega/2)}\right)^{n+1} \\ &= \int_M e^{\omega} \left(\frac{\omega}{1-e^{-\omega}}\right)^{n+1} = n+1,\end{aligned}$$

using again the calculation from earlier of the coefficient in the power series expansion. Then the following lemma, whose proof we postpone, gives that M is biholomorphic to $\mathbb{CP}^n$.

Lemma 2.5 (Theorem 1.1 in [7]). *If M is a compact Kähler manifold and L is a positive line bundle on M with $\int_M c_1^n(L) = 1$ and $\dim H^0(M,L) = n+1$ then M is biholomorphic to $\mathbb{CP}^n$.*

We can then assume that n is even (so $n \geq 2$) and that $c_1(M) = -(n+1)[\omega]$, which says that K_M is positive. By a theorem due independently to Yau [15] and Aubin [1] we know that M then admits a unique Kähler-Einstein metric with constant Ricci curvature equal to -1, that is a Kähler metric ω_{KE} such that

$$\mathrm{Ric}(\omega_{\mathrm{KE}}) = -\omega_{\mathrm{KE}}. \tag{2.3}$$

Recall here that the Riemann curvature tensor of a Kähler metric $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ in local holomorphic coordinates has components given by

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}^j},$$

the Ricci curvature tensor is its trace

$$R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 \log\det(g_{k\bar{\ell}})}{\partial z^i \partial \bar{z}^j},$$

and the Ricci form is defined by

$$\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

so that the Kähler-Einstein condition (2.3) is equivalent to

$$R_{i\bar{j}} = -g_{i\bar{j}}.$$

With this in mind, we have the following:

Lemma 2.6. *If (M,ω) is a Kähler-Einstein manifold of complex dimension $n\geq 2$, so that $\mathrm{Ric}(\omega)=\lambda\omega$ for some $\lambda\in\mathbb{R}$, then we have*

$$\left(\frac{2(n+1)}{n}c_2(M)-c_1^2(M)\right)\cdot[\omega]^{n-2}\geq 0, \tag{2.4}$$

with equality iff ω has constant holomorphic sectional curvature.

Proof. The tensor

$$R^0_{i\bar{j}k\bar{\ell}}=R_{i\bar{j}k\bar{\ell}}-\frac{\lambda}{n+1}(g_{i\bar{j}}g_{k\bar{\ell}}+g_{i\bar{\ell}}g_{k\bar{j}})$$

vanishes iff ω has constant holomorphic sectional curvature (see e.g. [6, Proposition IX.7.6]). Its tensorial norm square is easily computed as

$$\begin{aligned}|\mathrm{Rm}^0|^2&=g^{i\bar{q}}g^{p\bar{j}}g^{k\bar{s}}g^{r\bar{\ell}}R^0_{i\bar{j}k\bar{\ell}}R^0_{p\bar{q}r\bar{s}}\\&=|\mathrm{Rm}|^2+\frac{\lambda^2}{(n+1)^2}g^{i\bar{q}}g^{p\bar{j}}g^{k\bar{s}}g^{r\bar{\ell}}(g_{i\bar{j}}g_{k\bar{\ell}}+g_{i\bar{\ell}}g_{k\bar{j}})(g_{p\bar{q}}g_{r\bar{s}}+g_{p\bar{s}}g_{r\bar{q}})\\&\quad-\frac{2\lambda}{n+1}g^{i\bar{q}}g^{p\bar{j}}g^{k\bar{s}}g^{r\bar{\ell}}(g_{i\bar{j}}g_{k\bar{\ell}}+g_{i\bar{\ell}}g_{k\bar{j}})R_{p\bar{q}r\bar{s}}\\&=|\mathrm{Rm}|^2+\frac{\lambda^2}{(n+1)^2}(2n^2+2n)-\frac{4\lambda}{n+1}R,\end{aligned}$$

where R denotes the scalar curvature. The assumption $R_{i\bar{j}}=\lambda g_{i\bar{j}}$ gives $R=\lambda n$ and $|\mathrm{Ric}|^2=\lambda^2 n$. Then

$$|\mathrm{Rm}^0|^2=|\mathrm{Rm}|^2-\frac{2\lambda^2 n}{n+1}.$$

On the other hand if $\Omega_i^j=\sqrt{-1}R^j_{ik\bar{\ell}}dz^k\wedge d\bar{z}^\ell$ denote the curvature forms, then Chern-Weil theory says that

$$\frac{1}{2\pi}\mathrm{Ric}(\omega)=\frac{1}{2\pi}\sum_i\Omega_i^i=\frac{\sqrt{-1}}{2\pi}R_{k\bar{\ell}}dz^k\wedge d\bar{z}^\ell,$$

is a closed form that represents $c_1(M)$ in $H^2(M,\mathbb{R})$, while the form

$$\frac{1}{4\pi^2}\mathrm{tr}(\Omega\wedge\Omega)=\frac{1}{4\pi^2}\sum_{k,i}\Omega_i^k\wedge\Omega_k^i=\frac{(\sqrt{-1})^2}{4\pi^2}\sum_{k,i}R^k_{ip\bar{q}}R^i_{kr\bar{s}}dz^p\wedge d\bar{z}^q\wedge dz^r\wedge d\bar{z}^s,$$

represents $c_1^2(M)-2c_2(M)$. Since (2.4) is an integral inequality, we can ignore torsion in integral cohomology, and so we can use Chern-Weil forms to prove (2.4). Given a point $p\in M$ we choose local holomorphic coordinates so that p we have $g_{i\bar{j}}=\delta_{ij}$, and so also

$$\begin{aligned}\omega^n&=n!(\sqrt{-1})^n dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n,\\\omega^{n-2}&=(n-2)!(\sqrt{-1})^{n-2}\sum_{i<j}dz^1\wedge d\bar{z}^1\wedge\cdots\wedge\widehat{dz^i\wedge d\bar{z}^i}\wedge\cdots\\&\qquad\cdots\wedge\widehat{dz^j\wedge d\bar{z}^j}\wedge\cdots\wedge dz^n\wedge d\bar{z}^n,\end{aligned}$$

and it follows that at p we have

$$\begin{aligned} n(n-1)\mathrm{tr}(\Omega\wedge\Omega)\wedge\omega^{n-2} &= \sum_{k,i}\sum_{p\neq r}(R^k_{ip\overline{p}}R^i_{kr\overline{r}} - R^k_{ip\overline{r}}R^i_{kr\overline{p}})\omega^n \\ &= \sum_{k,i,p,r}(R^k_{ip\overline{p}}R^i_{kr\overline{r}} - R^k_{ip\overline{r}}R^i_{kr\overline{p}})\omega^n \\ &= (|\mathrm{Ric}|^2 - |\mathrm{Rm}|^2)\omega^n = (\lambda^2 n - |\mathrm{Rm}|^2)\omega^n. \end{aligned}$$

Hence this holds at all points, and so

$$\begin{aligned} |\mathrm{Rm}^0|^2\frac{\omega^n}{n(n-1)} &= -\mathrm{tr}(\Omega\wedge\Omega)\wedge\omega^{n-2} + \lambda^2\left(\frac{1}{n-1} - \frac{2}{(n+1)(n-1)}\right) \\ &= -\mathrm{tr}(\Omega\wedge\Omega)\wedge\omega^{n-2} + \frac{\lambda^2}{n+1}. \end{aligned}$$

Now notice that

$$\frac{1}{4\pi^2}\int_M \lambda^2\omega^n = \frac{1}{4\pi^2}\int_M (\lambda\omega)^2\wedge\omega^{n-2} = c_1^2(M)\cdot[\omega]^{n-2},$$

and so

$$\frac{1}{n(n-1)4\pi^2}\int_M |\mathrm{Rm}^0|^2\omega^n = \left(2c_2(M) - \left(1-\frac{1}{n+1}\right)c_1^2(M)\right)\cdot[\omega]^{n-2},$$

which implies what we want. □

We claim that equality in (2.4) does in fact hold in our case. This will finish the proof of Theorem 1.1, since then M would have constant negative holomorphic sectional curvature, and since it is also simply connected it would be biholomorphic to the unit ball in $\mathbb{C}^n$ (see e.g. [6, Theorem IX.7.9]), which is impossible.

We already know that $c_1^2(M) = (n+1)^2[\omega^2]$. To compute $c_2(M)$ we notice that by definition $p_1(M) = p_1(TM) = -c_2(TM\otimes\mathbb{C})$. But $TM\otimes\mathbb{C}\cong TM\oplus\overline{TM}$ and the Chern classes satisfy $c_k(\overline{TM}) = (-1)^k c_k(TM)$, so

(2.5)
$$\begin{aligned} p_1(M) &= -c_2(TM\oplus\overline{TM}) = -c_2(TM) - c_2(\overline{TM}) - c_1(TM)\cdot c_1(\overline{TM}) \\ &= -2c_2(M) + c_1^2(M). \end{aligned}$$

Putting this together with (2.2) we get

$$2c_2(M) = (n+1)^2[\omega^2] - (n+1)[\omega^2] = n(n+1)[\omega^2],$$

and thus equality holds in (2.4). This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let us denote by $\tau(M)$ the signature of M, which is the difference between the number of positive and negative eigenvalues for the intersection form

$$H_2(M,\mathbb{Z})\times H_2(M,\mathbb{Z})\to\mathbb{Z}.$$

The signature is a topological invariant (up to sign), and so

$$\tau(M) = \pm\tau(\mathbb{CP}^2) = \pm 1.$$

Hirzebruch's Signature Theorem [5, p.235] gives

$$\tau(M) = \frac{1}{3}\int_M p_1(M).$$

But from (2.5) we get

$$\frac{1}{3}\int_M (c_1^2(M) - 2c_2(M)) = \pm 1,$$

and Chern-Gauss-Bonnet's Theorem [5, p.235] gives

$$\int_M c_2(M) = \chi(M) = \chi(\mathbb{CP}^2) = 3,$$

and so

$$\int_M c_1^2(M) = 3(2 \pm 1) > 0.$$

A theorem of Kodaira [8] then says that M is projective. As before we see that $\chi(M, \mathcal{O}) = 1$ and then Riemann-Roch (see [5, p.233]) gives

$$\chi(M, \mathcal{O}) = \frac{K_M^2 + \chi(M)}{12} = \frac{K_M^2 + 3}{12},$$

which gives $\int_M c_1^2(M) = K_M^2 = 9$ (so in fact $\tau(M) = 1$). Let ω be as before, then $c_1(M) = \lambda[\omega]$ for some $\lambda \in \mathbb{Z}$. Then we have that $\lambda = \pm 3$, and these are exactly the same cases as in Theorem 1.1. If $\lambda = 3$, we need to check that $\dim H^0(M, L) = 3$. But we have $K_M = -3L$ and $K_M \cdot L = -3$ so Riemann Roch [5, p.233] gives

$$\chi(M, L) = 1 + \frac{L^2 - K_M \cdot L}{2} = 3.$$

Serre duality and Kodaira vanishing give

$$H^1(M, L) \cong H^1(M, K_M - L) = 0,$$

because $K_M - L = -4L$ is negative, and also

$$H^2(M, L) \cong H^0(M, K_M - L) = 0.$$

So $\chi(M, L) = \dim H^0(M, L) = 3$. Then the proof continues as in Theorem 1.1. □

Proof of Lemma 2.5. Let $(\varphi_1, \dots, \varphi_{n+1})$ be a basis of $H^0(M, L)$ and let $D_j = \{\varphi_j = 0\}$ be the corresponding divisors (they are nonempty, because otherwise L would be trivial, and so it would have $\dim H^0(M, L) = 1$). Define $V_n = M$ and

$$V_{n-k} = D_1 \cap \dots \cap D_k$$

for $1 \le k \le n$.

Lemma 2.7. *For each $0 \le r \le n$ we have that*

(1) *V_{n-r} is irreducible, of dimension $n-r$ and Poincaré dual to $c_1^r(L)$*

(2) *The sequence*
$$0 \to \mathrm{Span}(\varphi_1, \dots, \varphi_r) \to H^0(M, L) \to H^0(V_{n-r}, L)$$
is exact, where the last map is given by restriction.

Proof. The proof is by induction on r, the case $r = 0$ being obvious. Assuming that (1) and (2) hold for $r-1$, we see that V_{n-r+1} is irreducible and that φ_r is not identically zero on it. Hence $V_{n-r} = \{x \in V_{n-r+1} \mid \varphi_r(x) = 0\}$ is an effective divisor on V_{n-r+1} and so it can be expressed as a sum of irreducible subvarieties of dimension $n-r$. Since $c_1^{r-1}(L)$ is dual to V_{n-r+1} and $c_1(L)$ is dual to D_r we see that $c_1^r(L)$ is dual to V_{n-r}. If V_{n-r} were reducible, then $V_{n-r} = V' + V''$ and so
$$\begin{aligned} 1 = \int_M c_1^n(L) &= \int_M c_1^r(L) \cdot c_1^{n-r}(L) = \int_{V_{n-r}} c_1^{n-r}(L) \\ &= \int_{V'} c_1^{n-r}(L) + \int_{V''} c_1^{n-r}(L). \end{aligned}$$
But since L is positive, the last two term are both positive integers, and this is a contradiction. Thus (1) is proved. As for (2), the restriction exact sequence
$$0 \to \mathcal{O}_{V_{n-r+1}} \to \mathcal{O}_{V_{n-r+1}}(L) \to \mathcal{O}_{V_{n-r}}(L) \to 0,$$
gives
$$0 \to H^0(V_{n-r+1}, \mathcal{O}) \to H^0(V_{n-r+1}, L) \to H^0(V_{n-r}, L),$$
where the first map is given by multiplication by φ_r. This means that the kernel of the restriction map $H^0(V_{n-r+1}, L) \to H^0(V_{n-r}, L)$ is spanned by φ_r. This together with the statement in (2) for $r-1$ proves (2) for r. □

Now we apply Lemma 2.7 with $r = n$ and see that V_0 is a single point and that φ_{n+1} does not vanish there. So given any point of M there is a section of L that does not vanish there (i.e. L is base-point-free). Then we can define a holomorphic map $f : M \to \mathbb{CP}^n$ by sending x to $\{\varphi \in H^0(M, L) \mid \varphi(x) = 0\}$. This is a hyperplane in $H^0(M, L) \cong \mathbb{C}^{n+1}$ and so gives a point in $\mathbb{CP}^n$. If $y \in \mathbb{CP}^n$ corresponds to a hyperplane, which is spanned by some sections $(\varphi_1, \dots, \varphi_n)$, then $f(x) = y$ iff $\varphi_1(x) = \dots = \varphi_n(x) = 0$. Again Lemma 2.7 with $r = n$ says that $x = V_0$ exists and is unique, and so f is a bijection. □

3. Closing remarks

As a partial generalization of Theorems 1.1 and 1.2, Libgober-Wood [10] proved that a compact Kähler manifold of complex dimension $n \leq 6$ which is homotopy equivalent to $\mathbb{CP}^n$ must be biholomorphic to it.

A natural question is whether the Kähler hypothesis is really necessary in Theorem 1.1, and so one can ask whether a compact complex manifold diffeomorphic to $\mathbb{CP}^n$ must be biholomorphic to it. This is a well-known open problem (see e.g. [10]), and it is known that if this is true when $n = 3$

then there is no complex manifold diffeomorphic to S^6 (another famous open problem, see e.g. [9]):

Proposition 3.1. *If there exists a compact complex manifold M diffeomorphic to S^6, then there exists a compact complex manifold $\tilde{M}$ diffeomorphic to $\mathbb{CP}^3$ but not biholomorphic to it.*

This well-known fact was remarked already in [2, p.223].

Proof. Let M be a compact complex manifold diffeomorphic to S^6, and let $\tilde{M}$ be its blowup at one point $p \in M$. This is a compact complex manifold which is diffeomorphic to the connected sum $S^6 \sharp \overline{\mathbb{CP}^3}$, see e.g. [5, Proposition 2.5.8]. This is of course diffeomorphic to $\overline{\mathbb{CP}^3}$, and so also to $\mathbb{CP}^3$ (in fact, it is even oriented-diffeomorphic to $\mathbb{CP}^3$, since this manifold has the explicit orientation-reversing diffeomorphism $[z_0 : \cdots : z_3] \mapsto [\overline{z_0} : \cdots : \overline{z_3}]$). So $\tilde{M}$ is diffeomorphic to $\mathbb{CP}^3$, and if $\tilde{M}$ was biholomorphic to $\mathbb{CP}^3$ we would have

$$\int_{\tilde{M}} c_1(\tilde{M})^3 = \int_{\mathbb{CP}^3} c_1(\mathbb{CP}^3)^3 = 64.$$

But if we let $\pi : \tilde{M} \to M$ be the blowup map and $E = \pi^{-1}(p)$ be its exceptional divisor (which is biholomorphic to $\mathbb{CP}^2$), then we have (see [5, Proposition 2.5.5])

$$c_1(\tilde{M}) = \pi^* c_1(M) - 2[E],$$

where $[E]$ denotes the Poincaré dual of E. Since $b_2(M) = 0$ we have $c_1(M) = 0$, and so

$$\int_{\tilde{M}} c_1(\tilde{M})^3 = -8 \int_M [E]^3 = -8 \int_E [E]^2 = -8 \int_{\mathbb{CP}^2} c_1(\mathcal{O}(-1))^2 = -8,$$

since $[E]|_E$ equals the first Chern class of the tautological bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^2$ (see [5, Corollary 2.5.6]). Therefore $\tilde{M}$ is not biholomorphic to $\mathbb{CP}^3$, as claimed. $\square$

References

[1] Aubin, T. *Équations du type Monge-Ampère sur les variétés kähleriennes compactes*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 3, Aiii, A119–A121.

[2] Hirzebruch, F. *Some problems on differentiable and complex manifolds*, Ann. of Math. (2) **60** (1954), 213–236.

[3] Hirzebruch, F. *Topological methods in algebraic geometry*, Springer-Verlag, Berlin, 1995.

[4] Hirzebruch, F., Kodaira, K. *On the complex projective spaces*, J. Math. Pures Appl. **36** (1957), 201–216.

[5] Huybrechts, D. *Complex geometry. An introduction*, Springer-Verlag, Berlin, 2005.

[6] Kobayashi, S., Nomizu, K. *Foundations of differential geometry, Vol. II*, John Wiley & Sons, 1969.

[7] Kobayashi, S., Ochiai, T. *Characterizations of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ. **13** (1973), 31–47.

[8] Kodaira, K. *On the structure of compact complex analytic surfaces. I*, Amer. J. Math. **86** (1964), 751–798.

[9] LeBrun, C. *Orthogonal complex structures on S^6*, Proc. Amer. Math. Soc. **101** (1987), no.1, 136–138.
[10] Libgober, A.S., Wood, J. W. *Uniqueness of the complex structure on Kähler manifolds of certain homotopy types*, J. Differential Geom. **32** (1990), no. 1, 139–154.
[11] Milnor, J.W., Stasheff, J.D. *Characteristic classes*, Princeton University Press, 1974.
[12] Novikov, S.P. *Rational Pontrjagin classes. Homeomorphism and homotopy type of closed manifolds. I*, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965), 1373–1388.
[13] Severi, F. *Some remarks on the topological characterization of algebraic surfaces*, 1954.
[14] Yau, S.-T. *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.
[15] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), 339–411.

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